

Effective construction of canonical Hom-diagrams for equations over torsion-free hyperbolic groups

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April 19, 2016

Abstract

We show that, given a finitely generated group G as the coordinate group of a finite system of equations over a torsion-free hyperbolic group Γ , there is an algorithm which constructs a canonical solution diagram by constructing canonical (corrective extensions of) Γ -NTQ-groups. These groups are toral relatively hyperbolic Γ -limit groups. The diagram encodes all homomorphisms from G to Γ as compositions of factorizations through Γ -limit quotients (constructed by defining their generators inside canonical NTQ groups) and canonical automorphisms induced on the freely indecomposable factors of these quotients by canonical automorphisms of the corresponding NTQ-subgroups. Additionally, we show that a group is a Γ -limit group if and only if it is an iterated generalized double over Γ .

1 Introduction

Given a group G , another group L is said to be *fully residually G* (or *discriminated by G*) if, given any finite subset $A \subset L$, $1 \notin A$, there exists a homomorphism $\phi : L \rightarrow G$, such that $\phi(a) \neq 1$ for all $a \in A$. The study of groups discriminated by various classes of groups has been ongoing since the early twentieth century, with increased activity often occurring as other equivalent characterizations of such groups, and applications in different fields, are found. For example, the development of algebraic geometry over groups allowed finitely generated groups discriminated by a free group F to be understood as coordinate groups of irreducible systems of equations over F . This helped enable solutions to Tarski's problems on the elementary theory of free groups, given independently by Kharlampovich-Myasnikov and Sela ([17] and [26]).

Throughout, we let Γ be an arbitrary fixed torsion-free hyperbolic group. While some of the theory of fully residually free groups can be generalized easily to groups discriminated by Γ , in general groups discriminated by Γ are more difficult to work with for several reasons. Unlike finitely generated fully residually free groups, finitely generated fully residually Γ groups may be not finitely presented and can contain finitely generated subgroups with no finite presentation, so algorithmic results are often be more difficult to obtain.

For a group G which is equationally Noetherian (i.e. every system of equations over G with n variables is equivalent to a finite sub-system), there are many descriptions of the class of finitely generated fully residually G groups. Let L be a finitely generated group and G an equationally Noetherian group. The following are equivalent:

1. L is a G -limit group (in the space of marked groups, see [9] for definition)
2. L is fully residually G
3. L embeds in an ultrapower of G
4. $Th_{\forall}(G) \subseteq Th_{\forall}(L)$ (Th_{\forall} is universal first order theory)
5. $Th_{\exists}(G) \supseteq Th_{\exists}(L)$ (Th_{\exists} is existential first order theory)
6. L is the coordinate group $G_{R(S)}$ (see Section 2.1) of an irreducible (in the Zariski topology) algebraic set over G defined by a system S of coefficient-free equations (call such a system an *irreducible system*)

There are analogous equivalent characterizations for the case with coefficients (see [18] Theorem B).

Given a group G , a free rank one extension of centralizer over G is a group $E = \langle G, t \mid [C_G(U), t] = 1 \rangle$, where $U \subseteq G$ is finite. Note that $E \cong G *_{C_G(U)} \cong G *_{C_G(U)} (C_G(U) \times \mathbb{Z})$. A free rank n extension of centralizer is defined similarly as $G *_{C_G(U)} (C_G(U) \times \mathbb{Z}^n)$. A group is called an *iterated extension of centralizer over G* if it may be obtained from G by a finite sequence of extension of centralizers (each centralizer in the previous group).

Proposition 1. (*Theorem E in [18]*) *Given a fixed toral relatively hyperbolic group G , a group L is a G -limit group if and only if it embeds into an iterated extension of centralizer over G .*

In fact, the embeddings into iterated extensions of centralizers are obtained via embeddings into G -NTQ groups, which are coordinate groups of nicely structured systems of equations (see Section 2.1.2). Some relevant details of such embeddings are described in Section 5, along with other related constructions.

Our first main theorem shows that, for a fixed torsion-free hyperbolic group Γ , the class of iterated generalized doubles over Γ (terminology introduced by Champetier and Guirardel in [3] for free groups) is also equivalent to the class of Γ -limit groups.

Definition 1. *A group G is a generalized double over a Γ -limit group L if it splits as $G = A *_C B$, or $G = A *_C$, where A and B are finitely generated such that:*

1. *C is a non-trivial abelian group whose images are maximal abelian subgroups in the vertex groups.*
2. *there is an epimorphism $\phi : G \rightarrow L$ which is injective on each vertex group*

Note that each vertex group is discriminated by Γ .

Definition 2. *A group is an iterated generalized double over Γ if it belongs to the smallest class of groups $\mathcal{IGD} - \Gamma$ that contains groups isomorphic to subgroups of Γ , and is stable under free products and the construction of generalized doubles over groups in $\mathcal{IGD} - \Gamma$.*

Champetier and Guirardel showed in [3] that $\mathcal{IGD} - F$ is the class of F -limit groups. Our first main theorem shows that the same holds for Γ -limit groups.

Theorem 1. *G is a Γ -limit group if and only if it is an iterated generalized double over Γ*

Theorem 1 is proved in Section 3. The proof uses Bass-Serre theory and the fact that Γ -limit groups are exactly the groups which embed into iterated extensions of centralizers over Γ .

Recall that the Grushko decomposition of a finitely generated group G is the free product decomposition $G = F_r * A_1 * \dots * A_k$, where F_r is a free group of finite rank, and each A_i is non-trivial, freely indecomposable, and not infinite cyclic (see [12]). This decomposition is unique up to permutation of the conjugacy classes of the A_i in G . In section 4.2, we describe a canonical decomposition of a certain type, called a primary JSJ decomposition, for a freely indecomposable Γ -limit group, which encodes all of its abelian splittings. Primary JSJ decompositions were shown to exist for Γ -limit groups in [27].

Our second main theorem gives an algorithm that, given a finite system of equations over Γ , constructs a finite diagram with associated groups which has certain canonical properties. Such a *canonical Hom-diagram* (which is described more precisely in Section 5.5) encodes all homomorphisms from the coordinate group $\Gamma_{R(S)}$ to Γ as compositions of factorizations through Γ -limit quotients, and certain canonical automorphisms which correspond to JSJ decompositions of freely indecomposable factors of those quotients. Furthermore certain *canonical NTQ groups* are associated to such a diagram (through a process also described in Section 5.5).

Theorem 2. *Let $S(Z, A) = 1$ be a finite system of equations over Γ . There is an algorithm to construct a complete set of canonical (corrective extensions of) Γ -NTQ groups that are, in particular, toral relatively hyperbolic Γ -limit groups, and associated completed canonical Hom-diagram and canonical Hom-diagram for $\Gamma_{R(S)}$. All solutions of $S(Z, A) = 1$ in Γ factor through the strict fundamental sequences corresponding to these (corrective extensions of) Γ -NTQ groups.*

NTQ Γ -limit groups N_i in the completed diagram with branches (7) are toral relatively hyperbolic and constructed by their finite presentations, and canonical groups of automorphisms are given by their generators (and we know their presentation).

Γ -limit groups G_i in the branch (6) of the Hom-diagram are constructed by defining their generators inside corresponding NTQ-groups N_i and canonical automorphisms of these Γ -limit groups are induced on their freely indecomposable factors by canonical automorphisms of N_i .

Theorem 2 is proved in Section 6.

We thank D. Groves and H. Wilton for constructing “cautionary examples” [11] that helped us to improve the exposition of this work, through necessary clarifications and error corrections.

2 Preliminaries

2.1 Algebraic geometry over groups

We start with some basic theory of algebraic geometry over groups. For more background, see [1]. Let G be a group generated by a finite set A and $F(X)$ the free group on $X = \{x_1, \dots, x_n\}$. For $S \subset G[X] = G * F(X)$, the expression $S(X, A) = 1$ is called a *system of equations* over G , and a *solution* of $S(X, A) = 1$ in G , is a G -homomorphism $\phi : G[X] \rightarrow G$ such that $\phi(S) = 1$ (a G -homomorphism is determined by $Z = \phi(X) \in G^n$; the notation $S(Z, A) = 1$ means that Z corresponds to a solution of S).

Denote the set of all solutions of $S(X, A) = 1$ in G by $V_G(S)$, the *algebraic set defined by S* . Define the Zariski topology on G^n by taking algebraic sets as a pre-basis of closed sets. Note that the algebraic set $V_G(S)$ uniquely corresponds to the normal subgroup $R(S) = \{T(X, A) \in G[X] \mid \forall Z \in G^n (S(Z, A) = 1 \rightarrow T(Z, A) = 1)\}$ in $G[X]$, called the *radical* of S . Call $G_{R(S)} = G[X]/R(S)$ the *coordinate group* of S . Every solution of $S(X, A) = 1$ in G can be described as a G -homomorphism $G_{R(S)} \rightarrow G$.

2.1.1 Quadratic equations

A system of equations $S(X, A) = 1$ is said to be (strictly) *quadratic* if each $x_i \in X$ that appears in S , appears at most (exactly) twice, where x_i^{-1} also counts as an appearance. There are four standard quadratic equations:

$$\prod_{i=1}^n [x_i, y_i] = 1; n \geq 1 \quad (1)$$

$$\prod_{i=1}^n [x_i, y_i] \prod_{i=1}^m c_i^{z_i} d = 1; n, m \geq 0, n + m \geq 1 \quad (2)$$

$$\prod_{i=1}^n x_i^2 = 1; n \geq 1 \quad (3)$$

$$\prod_{i=1}^n x_i^2 \prod_{i=1}^m c_i^{z_i} d = 1; n, m \geq 0, n + m \geq 1 \quad (4)$$

where c_i and d are non-trivial elements of G . Note that for any strictly quadratic word $S \in G[X]$, there is a G -automorphism of $G[X]$ that takes S to a standard quadratic word. Also, there is a surface with boundary associated to each standard quadratic equation, specifically an orientable surface of genus n with

zero punctures for (1), an orientable surface of genus n with $m + 1$ punctures for (2), a non-orientable surface of genus n and zero punctures for (3), and a non-orientable surface of genus n with $m + 1$ punctures for (4). For a standard quadratic equation S , let $\chi(S)$ be the Euler characteristic of the associated surface.

2.1.2 G -NTQ groups

General systems of equations can exhibit some properties similar to those of quadratic systems. This can be seen when systems are in a particular *quasi-quadratic* form.

Definition 3. *A system of equations $S(X, A) = 1$ over a group G generated by A , is called triangular quasi-quadratic over G or G -TQ, if it can be partitioned into subsystems:*

$$\begin{aligned} S_1(X_1, C_1) &= 1 \\ S_2(X_2, C_2) &= 1 \\ &\vdots \\ S_n(X_n, C_n) &= 1 \end{aligned}$$

where:

- (i) $\{X_1, \dots, X_n\}$ is a partition of X
- (ii) $G_i = G[X_i, \dots, X_n, A]/R_G(S_i, \dots, S_n)$ for $1 \leq i \leq n$
- (iii) $G_{n+1} = G * F$ (where F is a free group of finite rank) or a subgroup of this group that is a free product of F , G and conjugates of G by some generators of F .
- (iv) $C_i = X_{i+1} \cup \dots \cup X_n \cup A \subset G_{i+1}$ for $1 \leq i \leq n - 1$ and $C_n = A$.

Furthermore, for each i the subsystems S_i must have one of the following forms:

- (I) S_i is quadratic in X_i
- (II) $S_i = \{[x, y] = 1, [x, u] = 1 | x, y \in X_i, u \in U\}$ where U is a maximal cyclic subgroup of $F(X_{i+1}, \dots, X_n, A)$
- (III) $S_i = \{[x, y] = 1 | x, y \in X_i\}$
- (IV) S_i is empty

The number n is called the depth of the system.

Notice that it may be assumed that every subsystem of form (I) is actually a single quadratic equation in standard form. Also, it can be checked directly that $G_i \simeq G_{i+1}[X_i]/R_{G_{i+1}}(S_i)$. $S(X, A) = 1$ is called *non-degenerate triangular quasi-quadratic over G* or *G -NTQ* if it is G -TQ and for every i , the system $S_i(X_i, C_i) = 1$ has a solution in G_{i+1} , and if S_i is of form (II) the set U generates a centralizer in G_{i+1} . A *regular G -NTQ* system is a G -NTQ system in which each non-empty quadratic equation S_i is in standard form, and either $\chi(S_i) \leq -2$ and the quadratic equation has a non-commutative solution in G_{i+1} , or it is an equation of the form $[x, y]d = 1$ or $[x_1, y_1][x_2, y_2] = 1$.

Finally a group is called a (*regular*) *G -NTQ group* if it is isomorphic to the coordinate group of a (regular) G -NTQ system of equations. Note that in Sela's work [26], *ω -residually free towers* provide an analagous structure to F -NTQ groups, and in the work of Casals-Ruiz and Kazachkov [2], *graph towers* are, in a certain sense, higher dimensional analogues of NTQ systems for working with right angle Artin groups.

Suppose G is a toral relatively hyperbolic group. Every G -NTQ group is also toral relatively hyperbolic [18]. In certain circumstances, when the bottom level G_{n+1} is clear from the context we will be talking (abusing the language) about *NTQ* groups, not specifying G_{n+1} .

2.2 Graphs of groups

A *graph of groups* $\mathcal{G}(X)$ is a connected graph $X(V, E)$ labelled with a group G_v for each vertex $v \in V$, and a group G_e with monomorphisms $\alpha_e : G_e \rightarrow G_{\partial_0(e)}$, $\beta_e : G_e \rightarrow G_{\partial_1(e)}$ for each edge $e \in E$ ($\partial_0(e)$ and $\partial_1(e)$ denote the initial and terminal vertices of e respectively). Note that X is considered to be a non-oriented graph, (i.e. there is an involution $^- : E \rightarrow E$ with $\partial_0(e) = \partial_1(\bar{e})$ for each $e \in E$), so the monomorphisms of a graph of groups must also satisfy $\alpha_e(G_e) = \beta_{\bar{e}}(G_{\bar{e}})$.

Let T be a maximal subtree of X . The *fundamental group* of $X(V, E)$ with respect to T is the group $\pi(\mathcal{G}(X), T)$ which is generated by $\langle *_{v \in V} G_v, *_{e \in E} t_e \rangle$ with relations $\{t_e = 1 \forall e \in T, t_e t_{\bar{e}} = 1 \forall e \in E, t_{\bar{e}} \alpha(g) t_e = \beta(g) \forall g \in G_e, \forall e \in E\}$. Any choice of maximal subtree gives an isomorphic fundamental group of the graph of groups. A *splitting* of a group G over some class of group \mathcal{E} is an isomorphism from G to $\pi(\mathcal{G}(X(V, E)), T)$ where each G_e is in \mathcal{E} . Splittings are discussed in further detail in Section 4.

Graphs of groups are closely related to the Bass-Serre theory of groups acting without inversion on trees. We note here one significant consequence which will be used later.

Proposition 2. (see [4], Theorem 3.7 in [24]) *Given G the fundamental group of a graph of groups, and $H \leq G$. Then H is the fundamental group of the induced graph of groups with vertex and edge groups given by intersections of H with conjugates of vertex and edge groups of G , respectively.*

3 Iterated generalized doubles

We can now prove Theorem 1. We start with the following useful elementary example of a generalized double.

Lemma 1. *A free rank one extension of centralizer over Γ_1 , where $\Gamma_1 \leq \Gamma$, is a generalized double over Γ .*

Proof. Given $G = \langle \Gamma_1, t | [C_{\Gamma_1}(U), t] = 1 \rangle = \Gamma_1 *_{C_{\Gamma_1}(U)}$, let $\phi : G \rightarrow \Gamma_1$

$$\phi(x) = \begin{cases} x & ; x \in \Gamma_1 \\ 1 & ; x = t \end{cases}$$

ϕ is injective on Γ_1 and by definition a centralizer is maximal abelian in Γ_1 . \square

With Proposition 1, this is enough to prove one direction of Theorem 1.

Proposition 3. *Let G be a generalized double over a Γ -limit group L . Then G is a Γ -limit group.*

Proof. First consider the case where G is an amalgamated product. Let ϕ be the map from G to L as in the definition of generalized double. Denote the images of A, B, C under ϕ by A^ϕ, B^ϕ , and C^ϕ respectively. Let \hat{C} be the maximal abelian subgroup of L containing C^ϕ . Consider the group $\hat{L} = \langle L, t | [c, t] = 1, c \in \hat{C} \rangle$. Since L embeds into an iterated centralizer extension of Γ , so does \hat{L} . Let $\hat{G} = A^\phi *_{C^\phi} (B^\phi)^t$; clearly $\hat{G} \leq \hat{L}$. We claim that $G \cong \hat{G}$.

This can be showed using normal forms. In particular, the maps $\phi|_A : A \rightarrow \hat{G}$ and $\phi|_B : B \rightarrow \hat{G}$ defined by composing the restrictions of ϕ to A and B respectively with the natural embedding of A^ϕ and B^ϕ into \hat{G} (and in the case of $\phi|_B : B \rightarrow \hat{G}$, composing with conjugation by the stable letter t in between), are injective (only the conjugation need be checked, and this follows since t is stable letter for \hat{L}) by Theorem 1.6 of [24]. So define $\hat{\phi} : G \rightarrow \hat{G}$ by $a_1 b_1 \dots a_n b_n \mapsto a_1^\phi t^{-1} b_1^\phi t \dots a_n^\phi t^{-1} b_n^\phi t$ for each word in G in normal form ($a_i, b_i \notin C$). Since if $b_i \in C$ then $t^{-1} b_i^\phi t = b_i^\phi$ and $a_i \notin C$ so $a_i^\phi \notin \hat{C}$, we have $a_1^\phi t^{-1} b_1^\phi t \dots a_n^\phi t^{-1} b_n^\phi t$ in normal form. Since every element of \hat{G} has normal form of $a_1^\phi t^{-1} b_1^\phi t \dots a_n^\phi t^{-1} b_n^\phi t = \phi a_1 b_1 \dots a_n b_n$, ϕ is an isomorphism. Then G embeds into \hat{L} and so by Proposition 1, G is a Γ -limit group. The case for $G = A * C$ is similar, since $\hat{G} = A^\phi *_{C^\phi}$ (and using the fact that if images of C^ϕ don't coincide then they must be conjugate by commutative transitivity) again embeds into \hat{L} . \square

To prove the converse, we use the Bass-Serre tree.

Proposition 4. *Every Γ -limit group can be constructed by iterated generalized doubles over Γ .*

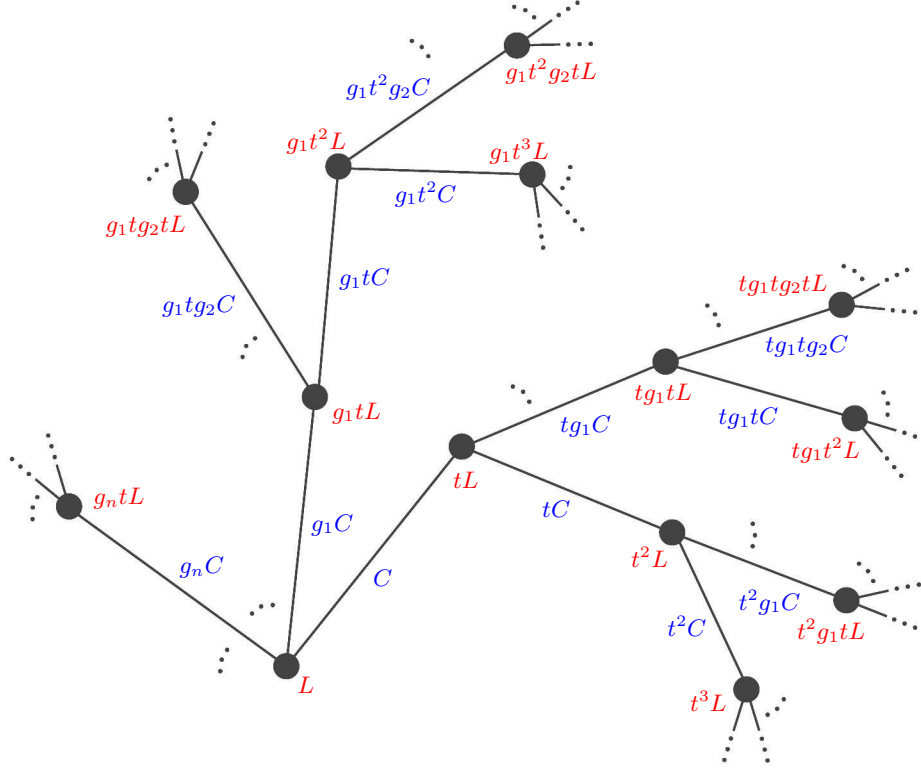


Figure 1: Bass-Serre tree

Proof. Given a freely indecomposable Γ -limit group, by Proposition 1, it may be embedded into an iterated extension of centralizers G over Γ . G is an iterated generalized double over Γ . We claim that every subgroup of G is also an iterated generalized double.

We proceed by induction. Let $H \leq G = \langle L, t | [C_L(u), t] = 1 \rangle$ for some Γ -limit group L and $u \in L$. Let $C = C_L(u)$. So G is an HNN extension $G = L *_C$. Now by Proposition 2, H acts (without inversions) on a tree with vertices given by the cosets gL (stabilizers are the groups $L^g \cap H$), and edges gC (corresponding to groups $C^g \cap H$), for each coset representative $g \in G$. In particular the initial vertex of the edge gC is gL and terminal vertex of gC is gtL .

Every coset gL has a representative of the form $t^{r_0} g_1 t^{r_1} \dots g_n t^{r_n}$ where the $g_i \in L$ are representatives for cosets of C in L , $g_n \neq 1$ and the r_i are non-zero except possibly r_0 and r_n . Now for each edge $t^{r_0} g_1 t^{r_1} \dots g_n t^{r_n} C$, the terminal vertex is $t^{r_0} g_1 t^{r_1} \dots g_n t^{r_n+1} L$ and the initial vertex is $t^{r_0} g_1 t^{r_1} \dots g_n t^{r_n} L$ if $r_n \neq 0$, and $t^{r_0} g_1 t^{r_1} \dots g_{n-1} t^{r_{n-1}} L$ if $r_n = 0$. See Figure 1 (vertex cosets are in red, edge cosets in blue). We want to show that the corresponding edge group

in the quotient graph of groups is maximal abelian in one of the corresponding vertex groups. Now $\text{Stab}(t^{r_0}g_1t^{r_1}\dots g_nt^{r_n}L) = H \cap L^{t^{-r_n}g_n^{-1}\dots g_1^{-1}t^{-r_0}}$ and $\text{Stab}(t^{r_0}g_1t^{r_1}\dots g_nt^{r_n}C) = H \cap C^{t^{-r_n}g_n^{-1}\dots g_1^{-1}t^{-r_0}}$.

Clearly $C^{t^{-r_n}g_n^{-1}\dots g_1^{-1}t^{-r_0}} \leq L^{t^{-r_n}g_n^{-1}\dots g_1^{-1}t^{-r_0}}$, and since $C^t = C$ we have $C^{t^{-r_n}g_n^{-1}\dots g_1^{-1}t^{-r_0}} = C^{t^{-r_{n-1}}g_{n-1}^{-1}\dots g_1^{-1}t^{-r_0}} \leq L^{t^{-r_{n-1}}g_{n-1}^{-1}\dots g_1^{-1}t^{-r_0}}$. Finally, $C^{g_n^{-1}t^{-r_{n-1}}\dots g_1^{-1}t^{-r_0}} \leq L^{g_n^{-1}t^{-r_{n-1}}\dots g_1^{-1}t^{-r_0}} = L^{t^{-r_{n-1}}\dots g_1^{-1}t^{-r_0}}$.

A conjugate of a centralizer is again a centralizer in the corresponding conjugate of the ambient group (this can be easily checked directly), and centralizers are maximal abelian. So intersecting a maximal abelian subgroup and its ambient group with H preserves maximality of the abelian subgroup (again, this can be easily checked directly).

Also, clearly every conjugate $L^{t^{-1}g^{-1}}$ is isomorphic to L , and so each $H \cap L^{t^{-1}g^{-1}}$ can be mapped monomorphically to L . Since any free factor of a Γ -limit group is a Γ -limit group, the proposition is proved. \square

Theorem 1 follows (since free products of Γ limit groups are also Γ -limit groups).

4 Further background on splittings

In this section we describe properties of JSJ decompositions of groups. We refer to [23], [7], [8], and [13] for further background on other notions of JSJ decompositions for groups.

Recall that an abelian splitting of a group G is an isomorphism to the fundamental group of a graph of groups with all abelian edge groups. It is often convenient to slightly abuse terminology and allow a splitting to refer to the graph of groups itself. An *elementary* splitting is one in which the graph of groups has exactly one edge. A splitting is *reduced* if the image of each edge group is a proper subgroup of the corresponding vertex group. A splitting of a group G is *essential* if it is reduced, all edge groups are abelian and for each edge group E , if $\gamma \in G, \gamma^k \in E$ for some $k \neq 0$, then $\gamma \in E$.

A subgroup $H \leq G$ is said to be *elliptic* with respect to a given splitting, if it is conjugate to a subgroup of some vertex group of the splitting. If H is not elliptic, then it is called *hyperbolic* with respect to the splitting.

There are certain elementary transformations of graphs of groups which preserve the fundamental group.

An *unfolding* of an elementary splitting $G \cong A *_C B$ is another splitting $G \cong A *_C B$ where C_1 is a proper subgroup of C and $B = C *_C B_1$. An *unfolding* of an elementary splitting $G \cong A *_C$ is another splitting $G \cong A_1 *_C$ where the image of C_1 in A_1 is a proper subgroup of C , and $A = A_1 *_C (tCt^{-1})$ where t is the stable letter of $A *_C$. We say a splitting is *unfolded* if there is no unfolding of any induced elementary splitting.

4.1 Classification of vertex groups

Vertex groups may be divided into three classes.

1. *Abelian* vertex groups
2. *Quadratically hanging* vertex groups. A vertex group G_v is quadratically hanging (or QH), if it admits either of the presentations (i.e. it is the fundamental group of a surface with finitely many punctures):

$$(i) \langle a_1, \dots, a_g, b_1, \dots, b_g | \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^m p_j \rangle \text{ where } g \geq 0, m \geq 1, \text{ and if } g = 0 \text{ then } m \geq 4$$

$$(ii) \langle a_1, \dots, a_g, p_1, \dots, p_m | \prod_{i=1}^g a_i^2 \prod_{j=1}^m p_j \rangle \text{ where } g \geq 1, m \geq 1$$

and furthermore:

- for each edge group G_e with $\partial_0(e) = v$, $\alpha_e(G_e)$ is conjugate to $\langle p_j \rangle$ in G_v for some $j = 1, \dots, m$
 - and for each $j = 1, \dots, m$ there is some edge e_j with $\partial_0(e_j) = v$ and $\alpha_{e_j}(G_{e_j})$ conjugate to $\langle p_j \rangle$ in G_v .
3. Vertex groups which are non-abelian and non-QH are called *rigid* vertex groups.

Any subgroup $H \leq G$ which is a QH vertex group in some splitting of G is called a *QH subgroup* of G . A QH subgroup $Q \leq G$ is a *maximal quadratically hanging subgroup* (or MQH subgroup), if given any elementary splitting of G with edge group G_e , either Q is elliptic in that splitting, or G_e can be conjugated into Q and that elementary splitting is induced by splitting Q along the image of G_e .

4.2 JSJ decompositions

There are various definitions of splittings which are called JSJ decompositions (after the topological notion of decomposing 3-manifolds along essential tori due to Jaco, Shalen, and Jacobsen) for various classes of groups. All of them are canonical in that they encode all other splittings (of a certain type) in some sense. We will use some particular qualifications for the canonical decompositions of Γ -limit groups.

Definition 4. *Given a reduced unfolded abelian splitting D of a freely indecomposable Γ -limit group L , call D a JSJ decomposition of L , if the following properties are satisfied:*

- (i) Every MQH subgroup of L is conjugate to a vertex group of the D , (so every QH subgroup of L can be conjugated into a vertex group of the JSJ), and every vertex group of the D which is not conjugate to a MQH subgroup of L is elliptic in any abelian subgroup of L .
- (ii) Any elementary abelian splitting of L which is hyperbolic in another elementary abelian splitting, can be obtained from D by the splitting of an MQH subgroup, which is induced by cutting the corresponding surface along an essential simple closed curve, and collapsing all other edges.
- (iii) Any elementary abelian splitting of L which is elliptic with respect to every other elementary abelian splitting of L , can be obtained from D by a sequence of collapsings, foldings, and conjugations.
- (iv) Any two reduced unfolded abelian splittings of L satisfying the above three properties can be obtained from one another by a sequence of slidings, conjugations, and modification of boundary monomorphisms by conjugations.
- (v) All non-cyclic abelian subgroups of L are elliptic in D .

In [27] (Theorem 1.10), every freely indecomposable Γ -limit group is shown to have such a JSJ decomposition.

5 Hom-diagrams and Γ -NTQ groups

We present here some constructions from [18] and [14], along with some similar background from [17] in the free group case. Building on those foundations, we define and prove the existence of canonical Hom-diagrams and associated canonical NTQ groups.

Unless otherwise stated, for systems with coefficients in Γ , all homomorphisms in Hom-diagrams for the system are assumed to be Γ -homomorphisms.

5.1 Canonical automorphisms

Consider an elementary abelian splitting of a group G . If we have $G = A *_C B$, for $c \in C$ define an automorphism $\phi_c : G \rightarrow G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(b) = b^c = c^{-1}bc$ for $b \in B$. If instead we have $G = A *_c$ then for $c \in C$ we define $\phi_c : G \rightarrow G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(t) = ct$.

Call the ϕ_c a *Dehn twist* obtained from the corresponding elementary abelian splitting of G . If G is an Γ -group, where Γ is a subgroup of one of the factors A or B , then Dehn twists that fix elements of the group $\Gamma \leq A$ are called *canonical Dehn twists*. Similarly, one can define canonical Dehn twists with respect to an arbitrary fixed subgroup K of G . Let $\mathcal{D}(G)$ [resp. $\mathcal{D}_\Gamma(G)$] be the set of all abelian splittings of G [resp. the set of all abelian splittings such that $\Gamma \leq G$ is elliptic].

Definition 5. Let $D \in \mathcal{D}(G)$ [$D \in \mathcal{D}_\Gamma(G)$] be an abelian splitting of a group G and G_v be either a QH or an abelian vertex of D . Then an automorphism $\psi \in \text{Aut}(G)$ is called a canonical automorphism corresponding to the vertex G_v if ψ satisfies the following conditions:

- (i) ψ fixes (up to conjugation) element-wise all vertex group in D , other than G_v . If $\Gamma \leq G_v$ then ψ also fixes each element of Γ . Note that ψ also fixes (up to conjugation) all the edge groups.
- (ii) If G_v is a QH vertex in D , then ψ is a Dehn twist (canonical Dehn twist) corresponding to some essential \mathbb{Z} -splitting of G along a cyclic subgroup of G_v .
- (iii) If G_v is an abelian subgroup then ψ acts as an automorphism on G_v which fixes all the edge subgroups of G_v .

Definition 6. Let e be an edge in an abelian splitting D of a group G . Let $\psi \in \text{Aut}(G)$. Call ψ a canonical automorphism corresponding to the edge e if ψ is a Dehn twist of G with respect to the elementary splitting of G along the edge e , induced from the splitting D . If $D \in \mathcal{D}_\Gamma(G)$, then ψ must fix Γ element wise.

Definition 7. The group of canonical automorphisms of a closed surface group G with respect to a trivial splitting D (ie. G is the only vertex group) is $A_D(G) = \text{MCG}(\Sigma)$ (the mapping class group of the surface).

Otherwise, the group of canonical automorphisms of a freely indecomposable group G with respect to an abelian splitting D is the subgroup $A_D(G) \leq \text{Aut}(G)$, generated by all canonical automorphisms of G corresponding to all edges, all QH vertices, and all abelian vertices of D . If G is not a Γ -group then include conjugation as well. The group of canonical automorphisms of a free product is the direct product of the canonical automorphism groups of factors.

5.2 Solution tree $\mathcal{T}(S, \Gamma)$

In this section we describe how to encode solutions to equations over Γ using a *Hom*-diagram. We begin by describing a *Hom*-diagram for generalized equations [16] over free groups. There is an algorithm described in [16] that, given a generalized equation $\Omega(X, A)$ over the free group $F = F(A)$, constructs a diagram, which encodes the solutions of Ω . Let G be the coordinate group $F_{R(\Omega)}$ of Ω considered as a system of equations over F . Specifically, the algorithm constructs a directed, finite, rooted tree T that has the following properties:

- (i) Each vertex v of T is labelled by a pair (G_v, Q_v) , where G_v is an F -quotient of G and Q_v the subgroup of canonical automorphisms in $\text{Aut}_F(G_v)$ corresponding to a splitting of G_v as a fundamental group of a graph of groups, that we find from the Elimination process of Ω . The root v_o is labelled by $(F_{R(\Omega)}, 1)$ and every leaf is labelled by $(F(Y) * F(A), 1)$ where Y is some finite set (called *free variables*). Each G_v , except possibly G_{v_o} , is fully residually F .

- (ii) Every (directed) edge $v \rightarrow v'$ is labelled by a proper surjective F -homomorphism $\pi(v, v') : G_v \rightarrow G_{v'}$.
- (iii) For every $\phi \in \text{Hom}_F(F_{R(\Omega)}, F)$, that is a solution of Ω (that must be non-cancellable in F) there is a path $p = v_0 v_1 \dots v_k$, where v_k is a leaf labelled by $(F(Y) * F(A), 1)$, elements $\sigma_i \in Q_{v_i}$, and a F -homomorphism $\phi_0 : F(Y) * F(A) \rightarrow F(A)$ such that

$$\phi = \pi(v_0, v_1)\sigma_1\pi(v_1, v_2)\sigma_2 \dots \pi(v_{k-2}, v_{k-1})\sigma_{k-1}\pi(v_{k-1}, v_k)\phi_0 \quad (5)$$

Considering all such F -homomorphisms ϕ_o , the family of the above sequences of homomorphisms is called the *fundamental sequence* over F corresponding to p .

Considering all such F -homomorphisms ϕ_o that produce solutions of Ω the family of the solutions of Ω factoring through the above fundamental sequence is called the *fundamental sequence for the generalized equation Ω* over F corresponding to p .

- (iv) The splitting of each fully residually free group G_v is its Grushko decomposition followed by the abelian splittings of the factors that are found by the Elimination process. If C_{vi} is such a factor, then the splitting is not necessarily the JSJ decomposition of C_{vi} , but it is maximal in a sense that it encodes all elementary abelian splittings of C_{vi} that can be found by the Elimination process, and has maximal QH and abelian vertex groups that can be found by the elimination process for G_v .

Notice that not all the homomorphisms that factor through (5) are solutions of Ω , but they all are homomorphisms from $F_{R(\Omega)}$ to F .

In [19] it is shown, given a finite system of equations $S(Z, A) = 1$ over Γ , how to construct a similar diagram encoding $\text{Hom}(\Gamma_{R(S)}, \Gamma)$. We briefly review the construction here.

First construct the generalized equations $S_1(X_1, A), \dots, S_n(X_n, A)$ over F from the system $S(Z, A) = 1$ over Γ by taking canonical representatives in $F(A)$ for elements of Γ (which exists by Theorem 4.5 of [22]), see Lemma 3 in [19] for more details on the construction of these systems. Construct the free group solution tree T_i as described above, for each generalized equation $S_i(X_i, A)$. We form a new, larger tree \mathcal{T} by taking a root vertex v_o labelled by $F(Z, A)$, attaching it to the root vertex of each T_i by an edge labelled by ρ_i , where $\rho_i : F(Z, A) \rightarrow F_{R(S_i)}$ is the homomorphism induced by canonical representatives. For each leaf v of T_i , labelled by $F(Y) * F$, build a new vertex w labelled by $F(Y) * \Gamma$ and an edge $v \rightarrow w$ labelled by the epimorphism $\pi_Y : F(Y) * F(A) \rightarrow F(Y) * \Gamma$ which is induced from $\pi : F \rightarrow \Gamma$ by acting as the identity on $F(Y)$. We call a path $b = v_0 v_1 \dots v_k$ from the root v_0 to a leaf v_k a *branch* b of \mathcal{T} .

Now associate to each branch b the set Φ_b consisting of all homomorphisms $F(Z, A) \rightarrow \Gamma$ of the form

$$\rho_i\pi(v_1, v_2)\sigma_2 \dots \pi(v_{k-2}, v_{k-1})\sigma_{k-1}\pi(v_{k-1}, v_k)\pi_Y\phi,$$

where

$$\rho_i \pi(v_1, v_2) \sigma_2 \cdots \pi(v_{k-2}, v_{k-1}) \sigma_{k-1} \pi(v_{k-1}, v_k)$$

is a solution of the generalized equation, and where $\sigma_j \in Q_{v_j}$ and $\phi \in \text{Hom}_\Gamma(F(Y) * \Gamma, \Gamma)$. Since $\text{Hom}_\Gamma(F(Y) * \Gamma, \Gamma)$ is in bijective correspondence with the set of functions Γ^Y , all elements of Φ_b can be effectively constructed.

Notice, that Φ_b is not a fundamental sequence over Γ , it is a fundamental sequence of solutions of a generalized equation over F , followed by a homomorphism $\pi_Y \phi$.

5.3 F -NTQ to Γ -NTQ reworking process

In this section we recall the construction that first was used in [14] and was developed in [19] that for each Φ_b constructs a strict fundamental sequence over Γ .

Definition 8. A fundamental sequence or a fundamental set of homomorphisms over Γ corresponding to the diagram

$$\Gamma_{R(S)} \rightarrow_{\pi_0} G_1 \rightarrow_{\pi_1} G_2 \rightarrow \cdots \rightarrow_{\pi_{n-1}} G_n = F * \Gamma * H_1 * \cdots * H_k$$

where

1. H_1, \dots, H_k are freely indecomposable groups isomorphic to subgroups of Γ , and G_1, \dots, G_n are Γ -limit groups.
2. $\pi_i, 0 < i < n-1$ are fixed proper epimorphisms, π_{n-1} is an epimorphism but may not be proper.
3. The homomorphisms in this sequence are compositions $\pi_0 \sigma_1 \pi_1 \dots \sigma_{n-1} \pi_{n-1} \tau$, where σ_i is a canonical automorphism of G_i corresponding to a Grushko decomposition of G_i followed by some abelian decompositions of the freely indecomposable factors where all non-cyclic abelian subgroups are elliptic. Canonical automorphisms are identical on the free factor of this Grushko decomposition.
4. τ is a homomorphism that maps each H_i monomorphically into a conjugate of a fixed subgroup of Γ (and for each H_i it is a fixed monomorphism followed by a conjugation) and maps F into Γ .

Definition 9. A fundamental sequence defined above is called strict if it has the following properties:

1. The image of each non-abelian vertex group of G_i under π_i is non-abelian.
2. For each $1 \leq i < n$, π_i is injective on rigid subgroups, edge groups, and subgroups generated by the images of edge groups in abelian vertex groups in G_{i-1} .

3. For each $1 \leq i < n$, if R is a rigid subgroup in G_i and $\{A_j\}$, $1 \leq j \leq m$, the abelian vertex groups in G_i connected to R by edge groups E_j with the maps $\eta_j : E_j \rightarrow A_j$, then π_i is injective on the subgroup $\langle R, \eta_1(E_1), \dots, \eta_m(E_m) \rangle$ which we will call the envelop of R .
4. The images of different factors in the Grushko decomposition of G_i under π_i are different factors in the free decomposition of G_{i+1} .

The construction of the tree of strict fundamental sequences over Γ , $\mathcal{T}(S, \Gamma)$ relies on a ‘reworking process’, which converts F -NTQ systems into appropriate Γ -NTQ systems. In $\mathcal{T}(S, \Gamma)$, each strict fundamental sequence $\bar{\Phi}_b$ corresponds to a Γ -NTQ group N_b (with depth of Γ -NTQ system equal to the length of the branch b) into which $\Gamma_{R(S)}$ maps by a homomorphism $\phi_b : \Gamma_{R(S)} \rightarrow N_b$. We briefly summarize the construction and important properties of $\mathcal{T}(S, \Gamma)$ here.

Each branch of \mathcal{T} has a corresponding F -NTQ system that comes from a generalized equation for canonical representatives. Considering the F -NTQ system as a system of equations over Γ does give groups through which all homomorphisms from $\Gamma_{R(S)}$ to Γ factor, but the systems may not be in Γ -NTQ form (as a result of the map $F \rightarrow \Gamma$ trivializing some vertex or edge groups). The reworking process fixes those degenerate portions, while maintaining the factoring of all homomorphisms $\Gamma_{R(S)} \rightarrow \Gamma$. For each F -NTQ system $N(X, A) = 1$, the reworking process constructs a new Γ -NTQ system of equations $L_1(X', A) = 1$ over Γ (see [14], section 3.3 and [19]). The following proposition is proved in [19], and it follows from the construction in [14].

Proposition 5. *Each branch of $\mathcal{T}(S, \Gamma)$ has the following form*

$$\Gamma_{R(S)} \rightarrow_{\psi_0} L_1 \rightarrow_{\psi_1} L_2 \rightarrow \dots \rightarrow_{\psi_{n-1}} L_n = F * \Gamma * K_1 * \dots * K_k,$$

where K_1, \dots, K_k are freely indecomposable groups isomorphic to freely indecomposable factors in the Grushko decomposition of Γ ; L_1, \dots, L_n are NTQ Γ -limit groups; ψ_0 is a fixed homomorphism; and ψ_i are fixed proper epimorphisms which are retractions onto L_{i+1} for $0 < i \leq n - 1$.

There is a strict fundamental sequence assigned to each branch. The homomorphisms in this sequence are compositions $\psi_0 \sigma_1 \psi_1 \dots \sigma_{n-1} \psi_{n-1} \tau$ where σ_i is a canonical automorphism corresponding to L_i , and τ is a homomorphism that maps each K_i monomorphically onto a conjugate of the corresponding subgroup of Γ (and for each K_i it is a fixed monomorphism followed by a conjugation), and maps F into Γ .

Every homomorphism from $\Gamma_{R(S)}$ to Γ factors through one of the fundamental sequences corresponding to the branches of $\mathcal{T}(S, \Gamma)$. Factors in the Grushko decomposition of L_i are mapped into different factors in the Grushko decomposition of L_{i+1} .

Each branch of $\mathcal{T}(S, \Gamma)$ can be effectively constructed.

5.4 Quasi-convex closure

Let Γ be a torsion-free non-elementary hyperbolic group, and H a subgroup of Γ given by generators. We describe a certain procedure that constructs a group

$K, H \leq K \leq \Gamma$ such that K is quasi-convex in Γ and every abelian splitting of K induces a splitting of H .

If H is hyperbolic in every cyclic splitting of Γ , then let $K = \Gamma$.

If H is elliptic in a cyclic splitting of Γ , we consider instead of Γ a vertex group Γ_1 containing a conjugate of H . Such a subgroup Γ_1 is quasi-convex and so hyperbolic, as it is a vertex group, and the quasi-convexity constants can be found effectively. If H is elliptic in a cyclic splitting of Γ_1 , continue this process until H is not elliptic in any decomposition of the corresponding vertex group Γ_j . So let K be the corresponding conjugate of Γ_j such that $H \leq K$. This subgroup K is a *quasi-convex closure* of H . K is relatively quasi-convex, so therefore hyperbolic. Note that hierarchical accessibility for hyperbolic groups was proved by Louder and Touikan in [20].

5.5 A complete set of canonical NTQ groups

In Section 5.2 we constructed a tree of strict fundamental sequences (or a *Hom*-diagram) encoding all solutions of a finite system $S(Z, A) = 1$ of equations over Γ using the tree of fundamental sequences for “covering” systems of equations over F . A *canonical Hom*-diagram is a tree of “canonical” strict fundamental sequences (in [17], Section 7.6, a similar tree was called the (augmented) canonical embedding tree $T_{CE}(F_{R(S)})$) which satisfy certain properties.

We define a canonical *Hom*-diagram iteratively. All the homomorphisms from $\Gamma_{R(S)}$ into Γ factor through a finite number of maximal Γ -limit quotients of $\Gamma_{R(S)}$. We continue with each maximal Γ -limit quotient (that we denote by G_1) in parallel. We factor G_1 into freely indecomposable factors G_{11}, \dots, G_{1t} and the free factor F_r . Homomorphisms from each G_{1i} to Γ in a canonical *Hom*-diagram are given by compositions of canonical automorphisms corresponding to a JSJ decomposition of G_{1i} with one of the finite number of fixed epimorphisms from G_{1i} onto its maximal standard (proper) Γ -limit quotients $G_{1i,1}, \dots, G_{1i,p}$ and with homomorphisms from these proper Γ -limit quotients to Γ that are constructed iteratively. Namely, we factor again each $G_{1i,j}$ into a free product of freely indecomposable factors and the free factor, etc. Since each proper sequence of Γ -limit quotients is finite, the construction of *Hom*-diagram terminates after finitely many steps. It terminates with either a fixed embedding of a Γ -limit group in the diagram into Γ followed by a conjugation in Γ and substitutions of the terminal free groups that appear in the diagram, into Γ .

The canonical *Hom*-diagram is a tree. For each branch of this tree

$$\Gamma_{R(S)} \rightarrow_{\pi_0} G_1 \rightarrow_{\pi_1} G_2 \rightarrow \dots \rightarrow_{\pi_{n-1}} G_n = F * \Gamma * H_1 * \dots * H_k, \quad (6)$$

there is a strict fundamental sequence assigned. Here

- 1) H_1, \dots, H_k are freely indecomposable groups isomorphic to subgroups of Γ ,
- 2) $\pi_i, 0 < i < n-1$ are fixed proper epimorphisms, π_{n-1} may not be proper.

3) The homomorphisms in this sequence are compositions $\pi_0\sigma_1\pi_1\ldots\sigma_{n-1}\pi_{n-1}\tau$, where σ_i is a canonical automorphism of G_i corresponding to a Grushko decomposition of G_i followed by the JSJ decompositions of the freely indecomposable factors,

4) τ is a homomorphism that maps each H_i monomorphically onto a conjugate of a fixed subgroup of Γ (and for each H_i it is a fixed monomorphism followed by a conjugation) and maps F into Γ .

The existence of such a diagram can be obtained from [27] and [14]. Indeed, the difference between the diagram that we described and the diagram constructed in [27] is that in [27] the homomorphism π_{n-1} must be proper and homomorphism τ can be an arbitrary embedding of groups H_1, \dots, H_k into Γ . In our diagram, if H_i has a non-trivial abelian splitting, then H_i appears as a factor in the Grushko decomposition of G_{n-1} , and in the leaves of the diagram we have a fixed (up to conjugacy) monomorphism of H_i into a conjugate of Γ . Notice that (non-canonical) fundamental sequences from Proposition 5 terminate with fixed conjugacy classes of monomorphisms of K_i into Γ , and we are going to use this fact to prove that canonical fundamental sequences satisfy the same property. Notice also that by [10], Lemma 7.2, H_i has infinitely many conjugacy classes of monomorphisms into Γ if and only if H_i has a non-trivial cyclic splitting. We define two embeddings of H_i into Γ to be equivalent if one is obtained from the other by pre-composing with an automorphism of H_i generated by Dehn twists corresponding to cyclic splittings of H_i and post-composing with a conjugation.

We claim that there is only a finite number of non-equivalent embeddings of each $H = H_i$ that appear in the diagram, into Γ . We will follow [10]. Suppose to the contrary, that there is an infinite family of non-equivalent monomorphisms. Let $\{h_i\}$ be a fixed system of generators of H . Consider a sequence $(\alpha_j : H \rightarrow \Gamma)$ of non-equivalent minimal monomorphisms (corresponding to minimal in the equivalence class sum of lengths of the images of $\{h_i\}$). After passing to a subsequence, the sequence can be taken to be stable, it converges to a stable isometric action of H on a real tree. Since $H = H/Ker_{\rightarrow}(\alpha_j)$, one can use Rips machine and the shortening argument to show that H has an essential cyclic splitting such that some of the monomorphisms in the sequence can be shorten by pre-composing with Dehn's twists corresponding to this splitting. Since the sequence consists of minimal monomorphisms, we have a contradiction with the assumption about infinite number of equivalence classes.

Construction of the NTQ group for a strict fundamental sequence.

We can assume (combining foldings and slidings) that all JSJ decompositions have the property that each vertex with non-cyclic abelian vertex group that is connected to a rigid subgroup is connected to only one rigid subgroup.

We assign an NTQ system to this branch as follows. First, replace each subgroup H_i that is not a hyperbolic closed surface group by its quasi-convex closure Γ_i , then G_n is replaced by $\hat{G}_n = F * \Gamma * \Gamma_1 * \dots * \Gamma_k$, $G_n \leq \hat{G}_n$. Notice that \hat{G}_n is a hyperbolic group. Let D_{n-1} be an abelian JSJ decomposition of G_{n-1} (we mean the decomposition into a free product of freely indecomposable factors followed by the JSJ decompositions of the factors). We order the edges

e_1, \dots, e_k between rigid subgroups. We extend the centralizer of the image of the edge group of e_1 in \hat{G}_n by a new letter, and obtain a new group $\hat{G}_n^{(1)}$, then iteratively for each $i \leq k$ we extend by a new letter the centralizer of the image of the edge group of e_i in $\hat{G}_n^{(i-1)}$ (see [17] for precise description).

Then the fundamental group of the subgraph of groups generated by the rigid subgroups in D_{n-1} will be embedded into this iterated centralizer extension $\tilde{G}_n = \hat{G}_n^{(k)}$ of \hat{G}_n . We also attach abelian vertex groups of D_{n-1} to \tilde{G}_n the following way. Consider edges (with maximal cyclic edge groups) that connect non-cyclic abelian vertex groups in D_{n-1} to a non-abelian non-QH vertex group. Some of the centralizers of the images of the edge groups may become conjugate in \tilde{G}_n . We join all edges with conjugate centralizers of the images of the edge groups into an equivalence class. For each equivalence class we do the following. Denote the edges in the class by $\bar{e}_1, \dots, \bar{e}_s$. Let m be the sum of the ranks of abelian vertex groups connected to $\bar{e}_1, \dots, \bar{e}_s$. We extend only the centralizer of the image of the edge group corresponding to \bar{e}_1 by new $m - s$ commuting letters (free rank $m - s$ extension of a centralizer defined in the introduction). Denote the obtained group by \ddot{G}_n . We attach QH subgroups of D_{n-1} identifying boundary components with their images in \tilde{G}_n , and obtain the new group \overline{G}_{n-1} such that G_{n-1} is embedded into \overline{G}_{n-1} . Notice that since G_{n-1} is a Γ -limit group, edge groups corresponding to edges adjacent to QH subgroups are maximal cyclic in QH subgroups.

The group \ddot{G}_n is a Γ -limit group as the iterated extension of centralizers of the Γ -limit group \hat{G}_n . The group \overline{G}_{n-1} is a Γ -limit group because it is a fundamental group of a family of regular quadratic equations over \ddot{G}_n . The group \overline{G}_{n-1} is an NTQ group by definition. It is also toral relatively hyperbolic by Dahmani's Combination Theorem 0.1, items (1) and (2) [5].

We denote \hat{G}_n by N_n and \overline{G}_{n-1} by N_{n-1} , then $G_{n-1} \leq N_{n-1}$.

Remark: Subgroups H_1, \dots, H_k may appear on different levels of the fundamental sequence as free factors in the Grushko decomposition of groups G_i . If H_s first time appears as a free factor of G_i (and not in G_{i-1}), then we consider its JSJ decomposition as part of the decomposition of G_i but not of G_{i+1}, \dots, G_{n-1} . So we will consider the restrictions of all the maps π_i, \dots, π_{n-1} on H_s as identical isomorphisms, and we will be extending centralizers of images of the edge groups of H_s in Γ_s only on level i .

Suppose we have already constructed the group N_i . We will show now how to construct N_{i-1} . Let D_{i-1} be an abelian JSJ decomposition of G_{i-1} (we mean the decomposition into a free product of freely indecomposable factors followed by the JSJ decompositions of the factors). For each freely indecomposable factor of G_{i-1} that is not a closed surface group and not a free abelian group we perform the construction in parallel and then take the free product of the constructed groups and all the factors that are closed surface groups and free abelian groups.

To simplify notation we now suppose that G_{i-1} is freely indecomposable and \bar{D}_{i-1} is its abelian JSJ decomposition. We order the edges e_1, \dots, e_k between rigid subgroups in \bar{D}_{i-1} . We freely extend the centralizer of the image of the

edge group of e_1 in N_i by a new letter, and obtain a new group $N_i^{(1)}$, then iteratively for each $j \leq k$ we freely extend by a new letter the centralizer of the image of the edge group of e_i in the previously constructed group $N_i^{(j-1)}$.

Then the fundamental group of the subgraph of groups generated by the envelopes of rigid subgroups in \bar{D}_{i-1} will be embedded into this iterated centralizer extension $\tilde{N}_i = N_i^{(k)}$ of N_i . We also attach abelian vertex groups of \bar{D}_{i-1} to \tilde{N}_i the following way. Consider edges that connect non-cyclic abelian vertex groups in \bar{D}_{n-1} to non-abelian non-QH vertex groups. Some centralizers of the images of the edge groups may become conjugate to centralizers of some other edge groups in \tilde{N}_i . We put two edges into the same equivalence class if some conjugates of the images of their edge groups in \tilde{N}_i commute. For each equivalence class we do the following. Denote the edges in the class by $\bar{e}_1, \dots, \bar{e}_s$. Let m be the sum of the ranks of abelian vertex groups connected to $\bar{e}_1, \dots, \bar{e}_s$, and p the sum of the ranks of their direct summands containing finite index subgroups generated by edge groups. We extend only the centralizer of the image in \tilde{N}_i of the edge group corresponding to \bar{e}_1 by new $m - p$ commuting letters. Denote the obtained group by \check{N}_i . We attach QH subgroups of \bar{D}_{n-1} to \check{N}_i identifying boundary components with their images in \check{N}_i that is a subgroup of \check{N}_i , and obtain the new toral relatively hyperbolic NTQ Γ -limit group N_{i-1} such that G_{i-1} is embedded into N_{i-1} .

We construct iteratively the group $N = N_1$, which is NTQ, and, therefore, a Γ -limit group, toral relatively hyperbolic and contains G_1 as a subgroup.

The set of all NTQ groups corresponding to a canonical Hom-diagram is a *complete set of canonical NTQ groups*. We often also consider a complete set of canonical Γ -NTQ groups, when the bottom level is a free product of F , Γ and several conjugates of freely indecomposable factors in the Grushko decomposition of Γ by new generators. Namely, in the beginning of the construction of the canonical NTQ group we can take this free product instead of $F * \Gamma * \Gamma_1 * \dots * \Gamma_k$, and then apply the construction.

Definition 10. *The group of canonical automorphisms of the NTQ group N_i ($i = 1, \dots, n - 1$) is the group of canonical automorphisms with respect to the Grushko decomposition of N_i followed by the abelian splitting such that MQH subgroups correspond to MQH subgroups of G_i abelian vertex groups correspond to the abelian vertex groups of N_i and non-QH non-abelian vertex groups are freely indecomposable factors in the Grushko decomposition of the group N_{i+1} .*

For each fundamental sequence (6) of the canonical Hom-diagram we assign the following fundamental sequence of the *completed* canonical Hom-diagram

$$\Gamma_{R(S)} \rightarrow_{\pi_0} N_1 \rightarrow_{\bar{\pi}_1} N_2 \rightarrow \dots \rightarrow_{\bar{\pi}_{n-1}} N_n = F * \Gamma * \Gamma_1 * \dots * \Gamma_k, \quad (7)$$

- 1) $\bar{\pi}_i, 0 < i \leq n - 1$ is a retraction on N_{i+1} ,
- 2) The homomorphisms in this *completed* fundamental sequence are compositions $\pi_0 \bar{\sigma}_1 \bar{\pi}_1 \dots \bar{\sigma}_{n-1} \bar{\pi}_{n-1} \tau$, where $\bar{\sigma}_i$ is a canonical automorphism of N_i ,
- 3) τ is a homomorphism that maps each Γ_i monomorphically onto a conjugate of a fixed subgroup of Γ (and for each Γ_i it is a fixed monomorphism followed by a conjugation) and maps F into Γ .

All Γ -homomorphisms from $\Gamma_{R(S)}$ to Γ that factor through the fundamental sequence (6) naturally factor through (7).

The set of all NTQ groups corresponding to a canonical *Hom*-diagram is a complete set of canonical NTQ groups. We often also consider a complete set of canonical Γ -NTQ groups, when the bottom level is a free product of F and several conjugates of Γ by new generators. Namely, in the beginning of the construction of the canonical NTQ group we can take this free product instead of $F * \Gamma * \Gamma_1 * \dots * \Gamma_k$, and then apply the construction.

Remark 1. *When considering fundamental sequences for NTQ groups (not necessary canonical NTQ groups) we only consider fundamental sequences such that canonical epimorphisms between the levels are retractions onto the lower level. They will be necessarily strict.*

An algorithm is given in the next section, which constructs a complete set of canonical Γ -NTQ groups (and systems) for $\Gamma_{R(S)}$ and the corresponding canonical *Hom*-diagram proving Theorem 2.

5.6 Implicit function theorem, formal solutions and generic families

Let $S(X, A) = 1$ be a system of equations with a solution in a group G . We say that a system of equations $T(X, Y, A) = 1$ is *compatible* with $S(X, A) = 1$ over G , if for every solution U of $S(X, A) = 1$ in G , the equation $T(U, Y, A) = 1$ also has a solution in G . More generally, a formula $\Phi(X, Y)$ is *compatible* with $S(X, A) = 1$ over G , if for every solution \bar{a} of $S(X, A) = 1$ in G there exists a tuple \bar{b} over G such that the formula $\Phi(\bar{a}, \bar{b})$ is true in G .

Suppose now that a formula $\Phi(X, Y, A)$ is compatible with $S(X, A) = 1$ over G . We say that $\Phi(X, Y, A)$ *admits a lift into a group L* such that $\Gamma_{R(S)} \leq L$, if the formula $\exists Y \Phi(X, Y, A)$ is true in L (here Y are variables and X are constants from $G_{R(S)}$). We say that the lift (is *effective* if there is an algorithm to decide for any formula $T(X, Y, A) = 1 \wedge W(X, Y, A) \neq 1$ whether the formula $T(X, Y, A) = 1 \wedge W(X, Y, A) \neq 1$ admits a lift into L , and if it does, to construct a solution in L .

Let N be an NTQ group. *Corrective extensions* of N are obtained by:

(i) Replacing each of the terminal factors H_i by a freely indecomposable quasi-convex subgroup Γ_i of Γ , where $H_i \leq \Gamma_i$ and the fixed monomorphism from H_i to the conjugate of Γ is extended to the fixed monomorphism from Γ_i to this conjugate of Γ .

(ii) Replacing each of the free abelian groups that appear in the Grushko decompositions on different levels of the NTQ group by a free abelian group of the same rank, that contains the original one as a subgroup of finite index.

Replacing each of the free abelian vertex groups that appear in the abelian decompositions of freely indecomposable factors on different levels of the NTQ group by a free abelian group of the same rank, that contains the original one as a subgroup of finite index.

(iii) Embedding the obtained group into the commutative transitive group using the procedure similar to the construction of the NTQ group for a strict fundamental sequence in Section 5.5. We do have a completed strict fundamental sequence corresponding to the NTQ system $S(X, A) = 1$ here, but since the subgroups at the base level were extended, we have to complete it again because centralizers of some abelian vertex groups that were not conjugate, may become conjugate.

We define the abelian size of N , denoted $ab(N)$, as the sum of the ranks of the abelian vertex groups in decompositions corresponding to different levels of N minus the sum of the ranks of their direct summands containing edge groups as subgroups of finite index. Then $ab(N)$ is the same as $ab(N_{corr})$ for each corrective extension (N_{corr}) of N .

Theorem 3. (*Implicit Function Theorem*) *Let $S(X, A) = 1$ be an NTQ system over a non-elementary torsion free hyperbolic group Γ . Suppose a formula $T(X, Y, A) = 1 \wedge W(X, Y, A) \neq 1$ is compatible with $S(X, Y) = 1$. Then this formula admits a lift into a family of NTQ groups N_1, \dots, N_k that are corrective extensions of $\Gamma_{R(S)}$, and toral relatively hyperbolic. Every solution from the fundamental sequence of solutions of $S(X, Y) = 1$ (see Remark 1) factors through the fundamental sequence for the NTQ system corresponding to one of the corrective extensions.*

This theorem is similar to the Parametrization Theorem, also called the Implicit function theorem ([16], Theorem 12) for free groups. A similar result is also formulated in [27], Theorem 2.3 for hyperbolic groups, but the formulation in [27] contains an error (see [19] for comments).

We now describe the construction of particular families of solutions, called *generic families*, of an NTQ system which imply nice lifting properties over that system. Consider a fundamental sequence with corresponding NTQ system $S(X, A) = 1$ of depth N . We construct generic families iteratively for each level k of the system, starting at $k = N$ and decreasing k . There is an abelian decomposition of G_k corresponding to the NTQ structure. Let $V_1^{(k)}, \dots, V_{M_k}^{(k)}$ be the vertex groups of this decomposition given some arbitrary order. We construct a generic family for level k , denoted $\Psi(k)$, by constructing generic families for each vertex group in order. We denote a generic family for the vertex group $V_i^{(k)}$ by $\Psi(V_i^{(k)})$. If there are no vertex groups, in other words the equation $S_k = 1$ is empty ($G_k = G_{k+1} * F(X_k)$) we take $\Psi(k)$ to be a sequence of growing different Merzljakov's words (defined in [16], Section 4.4). If the base group G is a free product $F(X_k) * \Gamma * H_1 * \dots * H_s$, where H_1, \dots, H_s are conjugates of subgroups of the torsion free hyperbolic group Γ , and there are fixed embedding $\mu_i : H_i \rightarrow \Gamma$, then the $\Psi(k)$ restricted to each H_i consists is $\mu_i(H_i)$ conjugated by a family of growing Merzliakov's words.

Remark 2. *When using generic families in this paper, by [27], Proposition 2.1, instead of a family of growing Merzliakov's words in Γ , one can just take $\mu_i(H_i)$ conjugated by a new letter, as well as one can take new letters for the basis of*

$F(X_k)$. So instead of a family of homomorphisms in G or Γ , we can consider generic family as a family of solutions into $G * F$ or $\Gamma * F$.

If $V_r^{(k)}$ is an abelian group then it corresponds to equations of the form $[x_i, x_j] = 1$ or $[x_i, u] = 1$, $1 \leq i, j \leq s$, where $u \in U$ runs through a generating set of a centralizer in G_{k+1} . A solution σ in G_{k+1} to equations of these forms is called *B-large* if there is some b_1, \dots, b_s with each $b_i > B$ such that $\sigma(x_i) = (\sigma(x_1))^{b_1 \dots b_i}$ or $\sigma(x_i) = u^{b_1 \dots b_i}$, for $1 \leq i \leq s$ (possibly renaming x_1). A generic family of solutions for an abelian subgroup $V_r^{(k)}$ is a family $\Psi(V_r^{(k)})$ such that for each B_i in any increasing sequence of positive integers $\{B_i\}_{i=1}^\infty$ there is a solution in $\Psi(V_r^{(k)})$ which is B_i -large.

If $V_r^{(k)}$ is a QH vertex group of this decomposition, let S be the surface associated to $V_r^{(k)}$. We associate two collections of non-homotopic, non-boundary parallel, simple closed curves $\{b_1, \dots, b_q\}$ and $\{d_1, \dots, d_t\}$. These collections should have the property that $S - \{b_1 \cup \dots \cup b_q\}$ is a disjoint union of three-punctured spheres and one-punctured Mobius bands, each of the curves d_i intersects at least one of the curves b_j non-trivially, and their union fills the surface S (meaning the collection $\{b_1, \dots, b_q, d_1, \dots, d_t\}$ have minimal number of intersections and $S - \{b_1 \cup \dots \cup b_q \cup d_1 \cup \dots \cup d_t\}$ is a union of topological disks).

Let β_1, \dots, β_q be automorphisms of $V_r^{(k)}$ that correspond to Dehn twists along b_1, \dots, b_q , and $\delta_1, \dots, \delta_t$ be automorphisms of $V_r^{(k)}$ that correspond to Dehn twists along d_1, \dots, d_t . We define iteratively a basic sequence of automorphisms $\{\gamma_{L,n}, \phi_{L,n}\}$ (compare with Section 7.1 of [16] where one particular basic sequence of automorphisms is used), which is determined by a sequence of $t + q$ -tuples $L = \{(p_{1,n}, \dots, p_{t,n}, m_{1,n}, \dots, m_{q,n})\}_{n=1}^\infty$

Let

$$\phi_{L,0} = 1$$

$$\gamma_{L,n} = \phi_{L,n-1} \delta_1^{m_{1,n}} \dots \delta_q^{m_{q,n}}, n \geq 1$$

$$\phi_{L,n} = \gamma_{L,n} \beta_1^{p_{1,n}} \dots \beta_t^{p_{t,n}}, n \geq 1$$

Assuming generic families have already been constructed for $V_i^{(k)}$, $i < r$, and for every vertex group in levels $k' > k$, and that Θ_k is a family of growing powers of Dehn twists for edges on level k , set $\Psi(k') = \Psi(V_{M_{k'}}^{(k')})\Theta_{k'}$ for $k < k' \leq N$ (in other words the generic family for level k' is the generic family of the last vertex group at that level) and set $\Psi(N+1) = \{1\}$. Let $\pi_k : G_k \rightarrow G_{k+1}$ be the canonical epimorphism. Let $\Sigma_r^{(k)} = \{\psi_1 \dots \psi_{r-1} | \psi_i \in \Psi(V_i^{(k)})\}$ be the collection of all compositions of generic solutions for previous vertex groups. We then say that

$$\Psi(V_r^{(k)}) = \{\mu_{L,n,\lambda_n} = \phi_{L,n} \delta_1^{\lambda_n} \dots \delta_q^{\lambda_n} \sigma_n \pi_k \tau | \sigma_n \in \Sigma_r^{(k)}, \tau \in \Psi(k+1)\}_{n=1}^\infty$$

where each λ_n is some positive integer, is a generic family for $V_r^{(k)}$ if it has the following property: Given any n and any tuple of positive numbers $\vec{A} =$

$(A_1, \dots, A_{nt+nq+1})$ with $A_i < A_j$ for $i < j$, Ψ contains a homomorphism μ_{n,L,λ_n} such that the tuple

$$\begin{aligned}\vec{L}_{n,r_n} &= (p_{1,1}, \dots, p_{t,1}, m_{1,2}, \dots, m_{q,2}, \dots, m_{1,n}, \dots, m_{q,n}, p_{1,n}, \dots, p_{t,n}, \lambda_n) \\ &= (L_1, \dots, L_{nt+nq+1})\end{aligned}$$

grows faster than \vec{A} in the sense that $L_1 \geq A_1$ and $L_{i+1} - L_i \geq A_{i+1} - A_i$.

Finally we set $\Psi(S) = \Psi(V_{M_1}^{(1)})$ to be a generic family of solutions for the G -NTQ system $S(X, A) = 1$. Notice that $\Psi(S)$ Γ -discriminates $\Gamma_{R(S)}$.

The following is a technical lemma that is used to prove Theorem 3.

Lemma 2. *If $\Psi(W)$ is a generic family of solutions for a regular NTQ system $W(X, A) = 1$, then for any formula $\Phi(X, Y, A) = U(X, Y, A) = 1 \wedge W(X, Y, A) \neq 1$ the following is true: if for any solution $\psi \in \Psi(W)$ there exists a solution of $\Phi(X^\psi, Y, A) = 1$, then Φ admits a lift into $\Gamma_{R(W)}$. If the NTQ-system $W(X, A) = 1$ is not regular, then for any such formula $\Phi(X, Y, A) = 1$ the following is true: if for any solution $\psi \in \Psi(W)$ there exists a solution of $\Phi(X^\psi, Y, A) = 1$, then Φ admits a lift into a family of corrective extensions of $\Gamma_{R(W)}$. There is a finite number of these extensions, and any solution of $W(X, A) = 1$ factors through one of them.*

Example 1. Consider the equation $[x, y] = [a, b]$ over a torsion-free hyperbolic group Γ . Define the following Γ -automorphisms of $\Gamma[X] = \Gamma * F(X)$:

$$\beta : x \rightarrow x, y \rightarrow xy$$

$$\delta : x \rightarrow yx, y \rightarrow y$$

Notice that big powers of these automorphisms produce big powers of elements. Let $n = 2a$ for some positive integer a , and define:

$$\phi_{p,n} = \delta^{m_1} \beta^{p_1} \dots \delta^{m_a} \beta^{p_a}$$

where $p = (p_1, m_1, \dots, p_a, m_a)$. Now take the solution τ of $[x, y] = [a, b]$ with $\tau(x) = a, \tau(y) = b$ and consider the family of mappings

$$\Psi = \{\mu_{p,n} = \phi_{p,n}\tau, p \in P_n\}_{n=1}^\infty$$

where for each n , P_n is an infinite set of n -tuples of large natural numbers. Then Ψ is a generic family of solutions in Γ for $[x, y] = [a, b]$.

Finally, if H is maximal Γ -limit quotient of $\Gamma_{R(S)}$ (there is no other Γ -limit quotient H_1 such that the map sending images of generators of $\Gamma_{R(S)}$ in H_1 into their images in H can be extended to a proper homomorphism), then since a discriminating family of homomorphisms for H factors through some branch b of $\mathcal{T}(S, \Gamma)$, there is some $\phi_b : \Gamma_{R(S)} \rightarrow N_b$ with $\phi_b(\Gamma_{R(S)}) = H$.

Each maximal Γ -limit quotient is isomorphic to $\Gamma_{R(S_1)}$, with $S_1 = 1$ an irreducible system, so there is a (canonical) fundamental sequence, with automorphisms on the top level corresponding to a JSJ decomposition of $\Gamma_{R(S_1)}$,

which factors through a Γ -NTQ groups N (as in Section 7 of [17]). This gives the canonical embeddings of each Γ -limit group L into some Γ -NTQ group, as mentioned in the introduction. We then define a generic family of homomorphisms for such a Γ -limit group $\Gamma_{R(S_1)}$ to be a generic family for a Γ -NTQ group N into which it embeds canonically in this manner.

The following is a simple but useful observation.

Lemma 3. *Let $S(Z, A) = 1$ be a finite system of equations over Γ . Let H be the image of the group $\Gamma_{R(S)}$ in a Γ -NTQ group N corresponding to a strict fundamental sequence in $\mathcal{T}(S, \Gamma)$. Then a generic family of Γ -homomorphisms from H to Γ corresponding to the JSJ decompositions of H and its images on all levels, factors through one of the branches of $\mathcal{T}(S, \Gamma)$.*

Proof. By construction of the solution tree for $S(Z, A) = 1$, there is some generic family of solutions for H . For this family, at each level of the solution tree, since there is finitely many branchings, some family factoring through at least one must be a generic family as well. \square

6 Constructing a complete set of (corrective extensions of) canonical Γ -NTQ groups

The proof of Theorem 2 is based on the observation that branches of the tree $\mathcal{T}(S, \Gamma)$ which contain generic families of homomorphisms for maximal Γ -limit quotients of $\Gamma_{R(S)}$, correspond to (almost) canonical NTQ systems.

6.1 Sol-maximal Γ -limit quotients

We define a partial order on the set of Γ -limit groups $\{H_{b_i} = \phi_{b_i}(\Gamma_{R(S)}) \leq N_{b_i}\}$ over all branches b_i of $\mathcal{T}(S, \Gamma)$, with N_{b_i} the Γ -NTQ group corresponding to b_i , as in Section 5.3. It's often notationally convenient to denote $H_{b_i}, N_{b_i}, \phi_{b_i}$ by H_i, N_i, ϕ_i . For given elements H_1, H_2 we say that $H_2 \leq_{\text{Sol}} H_1$ if for every homomorphism $\psi_2 : H_2 \rightarrow \Gamma$ that factors through N_2 there exists a homomorphism $\psi_1 : H_1 \rightarrow \Gamma$ that factors through N_1 such that $\phi_1\psi_1 = \phi_2\psi_2$. In this case the canonical map $\phi_2 : \Gamma_{R(S)} \rightarrow H_2$ can be split as $\tau\phi_1$, where ϕ_1 is the canonical homomorphism $\phi_1 : \Gamma_{R(S)} \rightarrow H_1$ and τ is a Γ -epimorphism from H_1 to H_2 .

Proposition 6. *There is an algorithm to find all homomorphisms $\phi_b : \Gamma_{R(S)} \rightarrow N_b$, where N_b is a Γ -NTQ group corresponding to a strict fundamental sequence in $\mathcal{T}(S, \Gamma)$ for a maximal Γ -limit quotient H of $\Gamma_{R(S)}$ such that $H = \phi_b(\Gamma_{R(S)})$.*

Proof. Let $\{\ell_1, \dots, \ell_n\}$ generate $\Gamma_{R(S)}$, and for each branch b in $\mathcal{T}(S, \Gamma)$, let $H_b = \phi_b(\Gamma_{R(S)})$ be the images of the homomorphisms into the corresponding Γ -NTQ groups. We proceed by considering the finite collection $\{H_b\}$, and the finite collection of pairs $\{(H_i, H_j)\}$.

First we describe an algorithm which tests a pair (H_1, H_2) if $H_2 \leq_{\text{Sol}} H_1$. Let $N_1(X, A) = 1$, $X = \{x_1, \dots, x_r\}$, and $N_2(Y, A) = 1$, $Y = \{y_1, \dots, y_s\}$, be

Γ -NTQ systems such that $N_1 = \Gamma_{R(N_1)}$ and $N_2 = \Gamma_{R(N_2)}$. There are words $\phi_1(\ell_i) = v_i(X, A) \in \Gamma_{R(N_1)}$ and $\phi_2(\ell_i) = \mu_i(Y, A) \in \Gamma_{R(N_2)}$ for $1 \leq i \leq n$.

Then the following first order formula is true in Γ , if and only if $H_2 \leq_{Sol} H_1$:

$$\forall h_1, \dots, h_s \exists g_1, \dots, g_r (N_2(h_1, \dots, h_s, A) = 1 \rightarrow (N_1(g_1, \dots, g_r, A) = 1 \wedge v_1(g_1, \dots, g_r, A) = \mu_1(h_1, \dots, h_s, A) \wedge \dots \wedge v_s(g_1, \dots, g_r, A) = \mu_s(h_1, \dots, h_s, A))).$$

In other words, for each Γ -homomorphism $\alpha : \Gamma_{R(N_2)} \rightarrow \Gamma$ there is a Γ -homomorphism $\beta : \Gamma_{R(N_1)} \rightarrow \Gamma$ such that $\beta_{H_1} = \tau \alpha_{H_2}$ (here α and β are restricted to H_2 and H_1).

Denote the above formula in Γ , as constructed from the pair H_1, H_2 , by $\Psi(H_1, H_2)$. We now describe an algorithm to test if $\Psi(H_1, H_2)$ is true. Every conjunction of equations is equivalent to some single equation, so let $S_\Psi = S_\Psi(h_1, \dots, h_s, g_1, \dots, g_r) = 1$ be an equation equivalent to $N_1(g_1, \dots, g_r, A) = 1 \wedge v_1(g_1, \dots, g_r, A) = \mu_1(h_1, \dots, h_s, A) \wedge \dots \wedge v_s(g_1, \dots, g_r, A) = \mu_s(h_1, \dots, h_s, A)$.

By Theorem 3, $\Psi(H_1, H_2)$ is true in Γ if and only if the corresponding equation S_Ψ (with h_1, \dots, h_s considered as coefficients in N_2^*) has a solution in each N_2^* , where N_2^* belongs to the (canonical) set of corrective extensions of N_2 . To obtain each corrective extension, for some abelian vertex groups A_i of N_2 , abelian groups \overline{A}_i with $A_i \leq_{f.i} \overline{A}_i$ are constructed exactly by adding particular roots to each A_i . Notice, that since N_1, N_2 are Γ -NTQ groups, the free products at their bases are free products of $\Gamma * F$ and some conjugates of factors in the Grushko decomposition of Γ , therefore a corrective extension does not extend the base group. However, Theorem 3 does not directly imply what those particular roots are. To find which roots to add, and construct the corrective extensions and check them for solutions, we use Dahmani's construction ([6]) of 'canonical representatives' for elements in N_2 (since it is toral relatively hyperbolic with parabolic subgroups given by the abelian vertex groups A_1, \dots, A_k of the NTQ structure) in the free product $F * A_1 * \dots * A_k$ (Theorem 4.4 in [6]). These canonical representatives give a disjunction of equations $S_\Psi^{(1)} \vee \dots \vee S_\Psi^{(N)}$, which is equivalent to a single equation S_Ψ^{can} , over $F * A_1 * \dots * A_k$.

Solutions for S_Ψ exist in N_2 if and only if solutions for S_Ψ^{can} exist in $F * A_1 * \dots * A_k$. Canonical representatives also exist for N_2^* (toral relatively hyperbolic as well), and using those, the equation S_Ω over N_2^* (with h_1, \dots, h_s considered as coefficients in N_2^*) induces the same equation S_Ω^{can} over a free product $F * \overline{A}_1 * \dots * \overline{A}_k$ with the same roots added to the same abelian subgroups. This is because Dahmani's construction of canonical representatives, applied to equivalent systems of triangular equations, give new equations corresponding to the "middles" of the triangles which, if the middles are "short" (as in hyperbolic case), are determined by S_Ω , and if the middles belong to parabolic subgroups, are just commuting equations. So similarly, solutions for S_Ω exist in N_2^* if and only if solutions for S_Ψ^{can} exist in $F * \overline{A}_1 * \dots * \overline{A}_k$. It is these free product extensions which can be found and checked algorithmically, since solving equations in

$F * \overline{A}_1 * \cdots * \overline{A}_k$ is just solving equations in each factor, and in the abelian factors, conventional linear algebraic methods of finding solutions determine the finite index extensions (by giving which roots of elements are needed for solutions). Then checking for a solution in $F * \overline{A}_1 * \cdots * \overline{A}_k$ is just checking in F , which can be done algorithmically by [21].

If $H_2 \leq_{Sol} H_1$ we remove H_2, N_2 and the corresponding branch from $\mathcal{T}(S, \Gamma)$. We now show that every non-Sol-maximal Γ -limit quotient is removed. If H_1 is not a Sol-maximal Γ -limit quotient, then there exists a Γ -NTQ group N_3 such that $H_1 \leq_{Sol} H_3$ and $\Psi(H_3, H_1)$. So H_1 is then removed from the collection $\{H_b\}$, and the proposition is proved. \square

Recall that the Grushko decomposition of a finitely generated group G is the free product decomposition $G = F_r * A_1 * \cdots * A_k$, where F_r is a free group of finite rank, and each A_i is non-trivial, freely indecomposable, and not infinite cyclic (see [12]). This decomposition is unique up to permutation of the conjugacy classes of the A_i in G .

Proposition 7. *Given a finite system of equations $S(Z, A) = 1$ over Γ , there is an algorithm which finds the Grushko decomposition of each such maximal Γ -limit quotient H .*

Proof. We show how to construct a free decomposition of the image H of $\phi_b : \Gamma_{R(S)} \rightarrow N_b$, for each strict fundamental sequence Φ_b . If H is a free product then there is a solution of the system $S(Z, A) = 1$ in $\Gamma * \Gamma$. Therefore we can solve the corresponding systems induced by canonical representatives of elements from $\Gamma * \Gamma$ in the group $F(A) * F(A)$. Then we will see the free product decomposition of the corresponding NTQ group over $\Gamma * \Gamma$ (since generators of each copy of Γ do not interact in the NTQ structure). The same NTQ group can be considered as an NTQ group over Γ if instead of the second copy of Γ we take Γ conjugated by a new element. Since H is a maximal Γ -limit quotient, the NTQ group corresponding to H must be freely decomposable, and we will obtain the induced free product decomposition of H (given by generators in the free factors of the NTQ group). \square

6.2 Corrective extensions of canonical Γ -NTQ groups

Proposition 8. *Suppose a branch b of the tree $\mathcal{T}(S, \Gamma)$ contains a generic family of homomorphisms such that the automorphisms for QH subgroups are constructed in [16], Sections 7.1, 7.2, for a canonical Γ -NTQ group N of a Sol-maximal Γ -limit quotient H of $\Gamma_{R(S)}$. Then the NTQ group N_b corresponding to the branch b is a free product of Γ -NTQ groups such that one of these Γ -NTQ groups M is a corrective extension of N .*

Proof. Since we know how to obtain a free decomposition of H , we can suppose now that H is freely indecomposable.

By Theorem 3, every solution of M in Γ can be obtained from a formula solution of M in a corrective extension N_{corr} of N (which is a retract on N)

composed with a solution of N_{corr} . All solutions of M in N_{corr} should be contained in a finite number of fundamental sequences over N_{corr} .

Let $M = M_1 > M_2 > \dots > M_n$ be a sequence of Γ -NTQ groups corresponding to different levels of M . The graph of groups decomposition Δ_1 corresponding to the top level may contain QH subgroups, abelian vertex groups and a free product of non-QH non-abelian vertex groups which is M_2 .

Lemma 4. *M does not have an abelian splitting where N and non-QH non-abelian vertex groups of Δ_1 are elliptic.*

Proof. All splittings of M are obtained from the Elimination process (which is a generalization of Makanin's process) for one of the generalized equations for canonical representatives. We denote by K the NTQ group constructed as a result of the Elimination process, such that M was re-worked from K . Then M cannot have a splitting containing a QH subgroup such that N is conjugate in the graph of groups corresponding to other vertex groups. Indeed, the QH subgroup has a pre-image QH subgroup in K and, therefore, big powers of Dehn twists corresponding to the generic family constructed in [16] are, actually, seen as big powers in canonical representatives, since they are (λ, μ) -quasi-geodesics. Hence there must be the induced splittings of H and, therefore, of N . Moreover, only some subwords of length bounded by μ of canonical representatives can become trivial in Γ , canonical representatives cannot contain unbounded powers of words that become trivial in Γ . Therefore there are no edges connected to QH subgroups in K that are mapped to the identity in the reworking process from K to M . Similarly, M cannot be a free extension of a centralizer of a group containing N because in solutions the new letters can be mapped into big powers that are much larger than the powers of the image of the corresponding centralizer in N . This, again will be seen in canonical representatives. \square

We recall a lemma from [15].

Lemma 5. *(Lemma 2.13 in [15]) If Q is a QH-subgroup of G and $H \leq G$, and not conjugate into the fundamental group of the graph of groups obtained by removing Q , then either $H \cap Q$ has finite index in Q and is a QH subgroup of H or H is a non-trivial free product.*

Therefore for each MQH subgroup Q of M , the intersection $N \cap Q$ is a finite index subgroup of Q , and there is a retract from Q to $N \cap Q$. This implies that $Q = N \cap Q$. Therefore MQH subgroups of N and M coincide.

On the second level, N and M may be freely decomposable, but will have the same number of free factors because their MQH subgroups are the same. Moreover, since the generalized equation for K was constructed from solutions corresponding to the generic family for N , the generalized equation corresponding to the second level of K can be transformed by a finite number of elementary transformations into a generalized equation on a disjoint family of intervals, each corresponding to a free factor. Going inductively from the top to the bottom we compare all levels of N and M . Their QH subgroups on all the levels are the same. If Δ_i is the abelian decomposition of a freely indecomposable factor P of

M_i on level i , then there is no abelian decomposition of P such that the corresponding freely indecomposable factor of N_i and non-QH nonabelian subgroups of M_i are elliptic. For each M_i there is a retraction from M_i onto N_i .

Notice, that since N is a Γ -NTQ group, the groups Γ_j that are conjugates of freely indecomposable factors in the Grushko decomposition of Γ at the bottom level of N are not extended in M , and the free products at the bottom of N and M are the same. This proves the proposition. \square

Recall that every Sol-maximal Γ -limit quotient H of $\Gamma_{R(S)}$ is isomorphic to the image of at least one homomorphism $\phi_b : \Gamma_{R(S)} \rightarrow N_b$, corresponding to a strict fundamental sequence in $\mathcal{T}(S, \Gamma)$, with notation as in Section 5.6. Moreover, in this case a generic family of solutions for a canonical Γ -NTQ group N of this Sol-maximal Γ -limit quotient H factors through N_b . But then a generic family of homomorphisms such that the automorphisms for QH subgroups are constructed in [16], Sections 7.1, 7.2 also factors through N_b . Therefore by Proposition 8 we constructed corrective extensions for all canonical Γ -NTQ groups for all Sol-maximal Γ -limit quotients of $\Gamma_{R(S)}$. Moreover, by construction, all solutions of $S(X, A) = 1$ that factor through canonical Γ -NTQ systems for Sol-maximal Γ -limit quotients, factor through the algorithmically constructed their corrective extensions.

Theorem 2 is proved.

Remark 3. *We can also replace corrective extensions of Γ -NTQ groups (when the bottom level is a free product of F , Γ and several conjugates of freely indecomposable factors in the Grushko decomposition of Γ by new generators) in the formulation of Theorem 2 by corrective extensions of NTQ Γ -limit groups (when the bottom level is a free product of F , Γ and several conjugates of freely indecomposable quasi-convex subgroups of Γ).*

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